

DISPLACEMENT INTERPOLATIONS FROM A HAMILTONIAN POINT OF VIEW

PAUL W.Y. LEE

ABSTRACT. One of the most well-known results in the theory of optimal transportation is the equivalence between the convexity of the entropy functional with respect to the Riemannian Wasserstein metric and the Ricci curvature lower bound of the underlying Riemannian manifold. There are also generalizations of this result to the Finsler manifolds and manifolds with a Ricci flow background. In this paper, we study displacement interpolations from the point of view of Hamiltonian systems and give a unifying approach to the above mentioned results.

1. INTRODUCTION

Due to its connections with numerous areas in mathematics, the theory of optimal transportation has gained much popularity in recent years. In this paper, we will focus on the, so called, displacement convexity in the theory of optimal transportation and its connections with Ricci curvature lower bounds and the Ricci flow.

Let M be a manifold. Recall that the optimal transportation problem corresponding to the cost function $c : M \times M \rightarrow \mathbb{R}$ is the following minimization problem:

Problem 1.1. Let μ and ν be two Borel probability measures on M . Minimize the following total cost

$$\int_M c(x, \varphi(x)) d\mu(x),$$

among all Borel maps φ which pushes μ forward to ν : $\varphi_*\mu = \nu$. Minimizers of this problem are called optimal maps.

In this paper, we are interested in cost functions defined by minimizing action functionals. More precisely, let $\mathbf{L} : \mathbb{R} \times TM \rightarrow \mathbb{R}$ be a smooth function, called Lagrangian. The function \mathbf{L} defines a family

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of cost functions $c_T : M \times M \rightarrow \mathbb{R}$ given by

$$(1) \quad c_T(x, y) = \inf_{\gamma(0)=x, \gamma(T)=y} \int_0^T \mathbf{L}(s, \gamma(s), \dot{\gamma}(s)) ds,$$

where the infimum is taken over all C^1 curves $\gamma : [0, T] \rightarrow M$ joining x and y : $\gamma(0) = x$ and $\gamma(T) = y$.

Assume that the Lagrangian \mathbf{L} is fibrewise strictly convex, super-linear, and the corresponding Hamiltonian flow is complete. Then the infimum in (1) is achieved. Moreover, if we also assume that the measure μ is absolutely continuous with respect to the Lebesgue class, then the optimal transportation problem corresponding to the cost function c_T has a minimizer φ_T which is unique μ -almost-everywhere [8, 23, 6, 15, 17].

As the time T varies, the optimal transportation problem defines a one parameter family of optimal maps φ_T . This, in turn, defines a one parameter family of probability measures $\mu_t := (\varphi_t)_* \mu$. These curves in the space of Borel probability measures, first introduced in [22, 23], are called McCann's displacement interpolations. In [9], displacement interpolations are called generalized geodesics and they were used to obtain a far reaching extension of the optimal transportation problem.

The connection between the optimal transportation problem and the Ricci curvature lies in the convexity of the classical entropy functional along displacement interpolations. More precisely, the entropy functional $\mathcal{E}_1 : \mathcal{P}_{ac} \rightarrow \mathbb{R}$ is defined on the space \mathcal{P}_{ac} of Borel probability measures which are absolutely continuous with respect to the Lebesgue class by

$$(2) \quad \mathcal{E}_1(\mu) = \int_M \rho(x) \log \rho(x) d\text{vol}(x)$$

where $\mu = \rho \text{ vol}$ and vol is the Riemannian volume form of g .

Let g be a Riemannian metric on the manifold M and let \mathbf{L} be the Lagrangian defined, in local coordinates, by

$$(3) \quad \mathbf{L}(x, v) = \frac{1}{2} \sum_{i,j=1}^n g_{ij} v_i v_j$$

where $g_{ij} = g(\partial_{x_i}, \partial_{x_j})$.

It was shown in [25, 12, 31] that the functional \mathcal{E}_1 is convex along any displacement interpolation defined by the cost function (1) with Lagrangian (3) if and only if the Ricci curvature defined by the Riemannian metric g is non-negative.

Instead of the entropy functional \mathcal{E}_1 , one can also consider the relative entropy \mathcal{E}_2 defined by

$$(4) \quad \mathcal{E}_2(\mu) = \int_M \rho(x) \log \rho(x) e^{-\mathfrak{U}(x)} d\mu(x),$$

where $\mathfrak{U} : M \rightarrow \mathbb{R}$ is a smooth function. It was shown in [25, 13, 29] that the entropy \mathcal{E}_2 is convex along any displacement interpolation defined by the cost function (1) with Lagrangian (3) if and only if the Bakry-Emery tensor $\text{Rc} + \nabla^2 \mathfrak{U}$ is non-negative.

The above results were also generalized to the Finsler case by [26]. In this case, (3) is replaced by Lagrangians $\mathbf{L} = \mathbf{L}(t, x, v)$ which are homogeneous of degree two in the v -variable. There are also various generalizations of the above results to manifolds with a Ricci flow background [24, 32, 20, 7].

The Euler-Lagrange equation of the action functional (1) is given by a Hamiltonian system on the cotangent bundle T^*M of the manifold M . In this paper, we study the displacement interpolations from the point of view of this Hamiltonian structure. By using the curvature of Hamiltonian systems introduced in [2], we prove a differential inequality (Theorem 2.2) which recover, unify, and generalize the above mentioned results on the Ricci curvature and the Ricci flow.

The structure of this paper is as follows. In section 2, the backgrounds and the statements of the main results in this paper are introduced. In section 3, we recall the definitions and properties of curvatures for Hamiltonian systems. In section 4, we recall several notions in the optimal transportation theory and its connections with Hamiltonian systems and the Hamilton-Jacobi equation. In section 5 and 6, we compute the curvature of some Hamiltonian systems studied in this paper. The rest of the sections are devoted to the proofs of the results.

2. MAIN RESULTS

In this section, we give the backgrounds and statements of the main results. For this, let us first discuss the minimization problem in (1). For simplicity, we assume that the manifold M is compact. In order to ensure the existence of minimizers in (1), we make the following assumptions on the Lagrangian \mathbf{L} throughout this paper without mentioning.

- \mathbf{L} is fibrewise uniformly convex (i.e. $v \mapsto \mathbf{L}(t, x, v)$ has positive definite Hessian for each time t in the interval $[0, T]$ and each point x in M),

- \mathbf{L} is super-linear (i.e. there is a Riemannian metric $|\cdot|$ and positive constants C_1 and C_2 such that $\mathbf{L}(t, x, v) \geq C_1|v| - C_2$ for all time t in $[0, T]$ and all tangent vector (x, v) in TM).

Under the above assumptions, the minimizers of (1) exist. Moreover, under additional assumption, they can be described as follows.

Let $\mathbf{H} : \mathbb{R} \times T^*M \rightarrow \mathbb{R}$ be the Hamiltonian defined by the Legendre transform of \mathbf{L}

$$\mathbf{H}(t, x, p) = \sup_{v \in T_x M} [p(v) - \mathbf{L}(t, x, v)].$$

Let $\vec{\mathbf{H}}$ be the Hamiltonian vector field of \mathbf{H} defined on the cotangent bundle T^*M by

$$\vec{\mathbf{H}} = \sum_{i=1}^n (\mathbf{H}_{p_i} \partial_{x_i} - \mathbf{H}_{x_i} \partial_{p_i}).$$

We also make the following assumption throughout this paper without mentioning.

- the time dependent flow $\Phi_{t,s}$ of $\vec{\mathbf{H}}$ defined by $\frac{d}{dt}\Phi_{t,s} = \vec{\mathbf{H}}(\Phi_{t,s})$ and $\Phi_{s,s}(x, p) = (x, p)$ is complete.

We also set $\Phi_t = \Phi_{t,0}$. Under the above assumptions, the minimizers of (1) are given by the projections $t \mapsto \pi(\Phi_t(x, p))$ of trajectories $t \mapsto \Phi_t(x, p)$ to the base M . Here $\pi : T^*M \rightarrow M$ is the natural projection. Moreover, if we assume that the measure μ is absolutely continuous with respect to the Lebesgue class, then Problem 1.1 with cost given by (1) also has a solution. More precisely, we have the following well-known result.

Theorem 2.1. [8, 23, 6] *Assume that M is a closed manifold and the measure μ is absolutely continuous with respect to the Lebesgue class. Then there is a solution φ_T , which is unique μ -almost-everywhere, to Problem 1.1 with cost c_T given by (1). Moreover, there is a locally semi-concave function \mathbf{f} such that*

$$\varphi_T(x) = \pi(\Phi_T(d\mathbf{f}_x)).$$

The function \mathbf{f} in Theorem 2.1 is the potential of the optimal map φ_T . For each such potential \mathbf{f} , one can define φ_t , $0 < t < T$, by

$$\varphi_t(x) = \pi(\Phi_t(d\mathbf{f}_x)).$$

Then φ_t is the optimal map between the measures μ and

$$(5) \quad \mu_t := (\varphi_t)_* \mu$$

for the cost c_t . The family of measures μ_t defined in (5) is called a displacement interpolation corresponding to the cost (1). It was first

introduced by [22, 23] in the Euclidean setting and it turns out to be the key to various connections of the optimal transportation problem with differential geometry. For instance, in the Riemannian case, the Ricci curvature of a manifold being non-negative is equivalent to the convexity of the entropy functional (2) along displacement interpolations corresponding to the cost function given by the square of the Riemannian distance.

Let \mathcal{P}_{ac} be the space of all Borel probability measures which are absolutely continuous with respect to the Lebesgue class. In this paper, we mainly focus on functionals of the following form. Let \mathbf{m}_t be a family of volume forms on M which vary smoothly in t . We will denote the measure induced by \mathbf{m}_t using the same symbol. Let us fix a function $\mathcal{F} : \mathbb{R} \rightarrow \mathbb{R}$. The measures \mathbf{m}_t and the function \mathcal{F} define a functional \mathbf{F} on the space $[0, T] \times \mathcal{P}_{ac}$ by

$$(6) \quad \mathbf{F}(t, \mu) = \int_M \mathcal{F}(\rho(t, x)) d\mu(x),$$

where $\mu = \rho(t, \cdot) \mathbf{m}_t$.

We show that the monotonicity and convexity properties of the functional defined in (6) are related to the curvature \mathbf{R} and the volume distortion v of the Hamiltonian system $\vec{\mathbf{H}}$. Here we give a brief introduction to these two concepts. A detail discussion can be found in Section 3 and 7.

Recall that the cotangent bundle T^*M of the manifold M is equipped with a symplectic form $\omega = \sum_{i=1}^n dp_i \wedge dx_i$ and the flow of the time-dependent Hamiltonian vector field $\vec{\mathbf{H}}$ is denoted by $\Phi_{t,s}$. Let $\pi : T^*M \rightarrow M$ be the natural projection and let \mathcal{V} be the vertical bundle defined as the kernel of the map $d\pi$. For each point (x, p) in the cotangent bundle T^*M of the manifold M , one can define a moving frame $E^{t,s} := (e_1^{t,s}, \dots, e_n^{t,s})^T$ and $F^{t,s} := (f_1^{t,s}, \dots, f_n^{t,s})^T$, called a canonical frame, of the symplectic vector space $T_{(x,p)}T^*M$ such that the following conditions are satisfied for each i and j :

- $e_i^{t,t}$ is contained in the vertical space $\mathcal{V}_{(x,p)}$ for each time t ,
- $\dot{e}_i^{t,s} = f_i^{t,s}$,
- $\ddot{e}_i^{t,s}$ is also contained in $\mathcal{V}_{(x,p)}$,
- the canonical frame $E^{t,s}, F^{t,s}$ is a Darboux basis for each (t, s) (i.e. $\omega(f_i^{t,s}, e_j^{t,s}) = \delta_{ij}$, $\omega(f_i^{t,s}, f_j^{t,s}) = \omega(e_i^{t,s}, e_j^{t,s}) = 0$).

Here and throughout this paper, we use the following notational convention: dot always denotes the derivative with respect to time t , s , or τ . For instance $\dot{V}^t = \frac{d}{dt}V^t$. In the case of the canonical frame, both $E^{t,s}$ and $F^{t,s}$ contain two time parameters t and s . In this case, dot

denotes the derivative with respect to the first parameter. For instance $\dot{e}_i^{t,s} = \frac{d}{dt}e_i^{t,s}$ and $\ddot{e}_i^{t,s} = \frac{d^2}{dt^2}e_i^{t,s}$.

The curvature operator $\mathbf{R}_{(x,p)}^t : \mathcal{V}_{(x,p)} \rightarrow \mathcal{V}_{(x,p)}$ of the Hamiltonian system $\vec{\mathbf{H}}$ is the linear operator defined by

$$\mathbf{R}_{(x,p)}^t(e_i^{t,t}) = -\ddot{e}_i^{t,t} = -\dot{f}_i^{t,t}.$$

Finally, the volume distortion $v : \mathbb{R} \times T^*M \rightarrow \mathbb{R}$ compares the volume forms \mathbf{m}_t with the frame $F^{t,t}$. More precisely,

$$v(t, x, p) := \log(\pi^* \mathbf{m}_t(f_1^{t,t}, \dots, f_n^{t,t})).$$

Curvature of Hamiltonian systems was first introduced and studied in [2]. It serves as a Hamiltonian analogue of the curvature operator in Riemannian geometry. More precisely, let us fix a Riemannian metric g and let \mathbf{H} be the kinetic energy given in a local coordinate chart by

$$(7) \quad \mathbf{H}(x, p) = \sum_{i,j=1}^n \frac{1}{2} g^{ij}(x) p_i p_j$$

where g^{ij} is the inverse matrix of $g_{ij} = g(\partial_{x_i}, \partial_{x_j})$. In this case, the curvature operator \mathbf{R}^t is independent of time t and it is, under some identifications using the Riemannian metric g , the curvature operator. Suppose that $\mathbf{m}_t = e^{-\mathfrak{U}} \text{vol}$, where $\mathfrak{U} : M \rightarrow \mathbb{R}$ is a smooth function and vol is the Riemannian volume form of g . Then the volume distortion is simply $v = -\pi^* \mathfrak{U}$.

Next, we consider a particular case of the functional \mathbf{F} with $\mathcal{F}(r) = \log(r)$. In this case, the functional \mathbf{F} is a time-dependent version of the relative entropy (4) and we obtain the following theorem which unifies several results in the literature.

Theorem 2.2. *Assume that $\mathcal{F}(r) = \log(r)$ and let b be any function of time t . Then the functional \mathbf{F} satisfies the following inequality:*

$$(8) \quad \begin{aligned} \frac{d^2}{dt^2} \mathbf{F}(t, \mu_t) + b(t) \frac{d}{dt} \mathbf{F}(t, \mu_t) + \frac{nb(t)^2}{4} &\geq \int_M \left(\text{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) \right. \\ &\quad \left. - \frac{d^2}{dt^2} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) - b(t) \frac{d}{dt} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \right) d\mu(\mathbf{x}) \end{aligned}$$

for each smooth displacement interpolation μ_t corresponding to the cost (1).

In Theorem 2.2, we assume that μ_t is a smooth displacement interpolation (see Section 4 for the definition). This allows us to focus on the ideas of the proof and avoids technical difficulties that arise due to the lack of regularity of the potential \mathbf{f} . These technicalities can be dealt with along the lines in [12, 32] and will be reported elsewhere.

By specializing Theorem 2.2 to the Riemannian case, we recover several well-known results. More precisely, let \mathbf{H} be the kinetic energy defined by (7). Let $b \equiv 0$ and $\mathbf{m}_t = e^{-\mathfrak{U}} \text{vol}$. Then the functional \mathbf{F} defined in (6) coincides with the relative entropy \mathcal{E}_2 defined in (4) and we recover the following result which appeared in [25, 13, 31, 29].

Corollary 2.3. *Let g be a Riemannian metric with Ricci curvature Rc . Then the Bakry-Emery tensor $Rc + \nabla^2 \mathfrak{U}$ satisfies the following condition*

$$(9) \quad Rc + \nabla^2 \mathfrak{U} \geq Kg$$

if and only if

$$(10) \quad \frac{d^2}{dt^2} \mathcal{E}_2(\mu_t) \geq \frac{2K}{T} C_T(\mu, \nu)$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Lagrangian defined by (3). Here $\nabla^2 \mathfrak{U}$ denotes the Hessian of the function \mathfrak{U} with respect to the given Riemannian metric g . $C_T(\mu, \nu)$ is defined by the optimal transportation problem

$$C_T(\mu, \nu) := \inf_{\varphi_* \mu = \nu} \int_M c_T(x, \varphi(x)) d\mu(x),$$

where the infimum is taken over all Borel maps φ which pushes μ forward to ν .

We remark that even though we replace the usual displacement interpolations by smooth displacement interpolations in Corollary 2.3, the equivalence between the Bakry-Emery condition (9) and the convexity of the functional \mathbf{F} still holds.

We also remark that the condition (10) does not involve any differential structure and it was used in [29, 21] to define Ricci curvature lower bound for more general metric measure spaces.

If we replace the Hamiltonian \mathbf{H} in (7) by a Finsler Hamiltonian (i.e. any Hamiltonian which is homogeneous of degree two in the p variable), then we recover the corresponding results in [26]. By considering functionals more general than (6), one can recover various other known results which are not covered by Theorem 2.2. We will not pursue this here. Instead, we consider Hamiltonians of the following form which motivated the whole work

$$(11) \quad \mathbf{H}(x, p) = \frac{1}{2} \sum_{i,j} g^{ij}(t, x) p_i p_j + U(t, x).$$

Here both the metric g and the potential U depend on time t .

We would like to apply Theorem 2.2 to this Hamiltonian and recover some results between the optimal transportation theory and the Ricci flow (c.f. [24, 32, 20, 7]). However, in all these works, a Hamiltonian \mathbf{H} is fixed and the functional \mathbf{F} is shown to have certain convexity properties along displacement interpolations corresponding to the cost function (1) with Hamiltonian \mathbf{H} . Instead we propose the following approach of using Theorem 2.2 to find Hamiltonians \mathbf{H} such that the functional \mathbf{F} is monotone or convex along displacement interpolations corresponding to \mathbf{H} .

First, we create some functional parameters to get rid of the undesire terms in (8). On the other hand, g should be related to the Ricci flow. Therefore, it is natural to pick the time dependent metric g satisfying the following equation:

$$(12) \quad \dot{g} = c_1 \text{Rc} + c_2 g,$$

where Rc is the Ricci curvature of the metric g at time t , c_1 and c_2 are functions depending only on time t which will be used to get rid of the undesire terms.

Note that the time dependence of various quantities in (12) is suppressed and this convention is used throughout this paper. Note also that the choice of g in (12) is natural from the point of view of the Ricci flow. In fact, if \bar{g} is a solution of the Ricci flow $\dot{\bar{g}} = -2\text{Rc}$, then

$$g(t, x) = a_1(t)\bar{g}(a_2(t), x)$$

satisfies (12).

We consider the following functional which is a special case of (6)

$$(13) \quad \mathcal{E}_3(t, \mu) = \int_M \log(\rho(t, x)) d\mu(x),$$

where $\mu = \rho(t, \cdot) \mathbf{m}_t$, $\mathbf{m}_t = e^{-k(t)u(t, \cdot)} \text{vol}$, u is the solution of the Hamilton-Jacobi equation

$$\dot{u} + \mathbf{H}(t, x, du) = 0$$

with initial condition $u|_{t=0} = \mathbf{f}$ (here \mathbf{f} is the potential which define the displacement interpolation μ_t), and k is another functional parameter.

Next, we choose suitable functions c_1 , c_2 , b , U , and measure \mathbf{m}_t such that the right hand side of (8) can be estimated. This leads to the following result which is a consequence of Theorem 2.2 (see the proof for some explanations on the choices of c_1 , c_2 , b , and U).

Corollary 2.4. *If the functions c_1, c_2, b, U satisfy*

$$c_1 k = -2, \quad \ddot{k} = -b\dot{k}, \quad 2\dot{k} = c_2 k - b k, \quad U = -\frac{c_1^2}{8} R,$$

then

$$\frac{d^2}{dt^2}\mathcal{E}_3(t, \mu_t) + b(t)\frac{d}{dt}\mathcal{E}_3(t, \mu_t) + \frac{n\dot{c}_2(t)}{2} + \frac{nc_2(t)^2}{4} + \frac{nb(t)^2}{4} \geq 0$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Hamiltonian \mathbf{H} given by (11) and (12). Here R denotes the scalar curvature of the given Riemannian metric g at time t .

If we further specialize to the case $k(t) = Ct^m$ and $m \neq 0$, then the rest of the functional parameters (c_1, c_2, b, U) are completely determined and Corollary 2.4 simplifies to the following result.

Corollary 2.5. *If $m \neq 0$, $k(t) = Ct^m$, $c_1(t) = -\frac{2}{Ct^m}$, $c_2(t) = \frac{m+1}{t}$, $b(t) = -\frac{m-1}{t}$, and $U(t, x) = -\frac{c_1(t)^2}{8}R(t, x)$, then*

$$\frac{d^2}{dt^2}\mathcal{E}_3(t, \mu_t) - \frac{m-1}{t}\frac{d}{dt}\mathcal{E}_3(t, \mu_t) \geq \frac{m(1-m)n}{2t^2}$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Hamiltonian \mathbf{H} given by (11) and (12).

If $k(t) \equiv C$ (i.e. $m = 0$), then Corollary 2.4 gives the following result.

Corollary 2.6. *If $k(t) \equiv C \neq 0$, $c_1(t) = -\frac{2}{C}$, $c_2(t) = b(t)$, and $U(t, x) = -\frac{1}{2C^2}R(t, x)$, then*

$$\frac{d^2}{dt^2}\mathcal{E}_3(t, \mu_t) + b(t)\frac{d}{dt}\mathcal{E}_3(t, \mu_t) + \frac{n\dot{b}(t)}{2} + \frac{nb(t)^2}{2} \geq 0$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Hamiltonian \mathbf{H} given by (11) and (12).

Corollary 2.5 and 2.6 contain some known results. If we set $m = -\frac{1}{2}$ and $C = -1$ in Corollary 2.5, then $c_1(t) = 2\sqrt{t}$ and $c_2(t) = \frac{1}{2t}$. A calculation shows that $g = \sqrt{t}\bar{g}$, where \bar{g} is a solution of the backward Ricci flow $\dot{\bar{g}} = 2\bar{\text{Rc}}$. Here $\bar{\text{Rc}}$ is the Ricci curvature of the Riemannian metric \bar{g} .

The Lagrangian \mathbf{L} is given by

$$\begin{aligned} \mathbf{L}(t, x, v) &= \frac{1}{2} \sum_{i,j} g_{ij}(t, x) v_i v_j + \frac{t}{2} R(t, x) \\ (14) \quad &= \frac{\sqrt{t}}{2} \left(\sum_{i,j} \bar{g}_{ij}(t, x) v_i v_j + \bar{R}(t, x) \right), \end{aligned}$$

where \bar{R} is the scalar of the metric \bar{g} . The corresponding action functional

$$\gamma \mapsto \int_0^T \mathbf{L}(t, \gamma(t), \dot{\gamma}(t)) dt$$

is Perelman's \mathcal{L} -functional introduced in [27].

Let $\overline{\text{vol}}$ be the Riemannian volume of the metric \bar{g} . Let \mathcal{E}_4 be the following functional

$$(15) \quad \mathcal{E}_4(t, \mu) = \int_M (\log(\rho(t, x)) + t^{-1/2} u(t, x)) d\mu(x) + \frac{n}{2} \log t,$$

where $\mu = \rho(t, \cdot) \overline{\text{vol}}$. The following result on \mathcal{E}_4 which appeared in [32, 20] is a special case of Corollary 2.5. It was also shown in [32, 20] how this result leads to the monotonicity of Perelman's reduced volume [27].

Corollary 2.7. [20, Theorem 1] *The functional \mathcal{E}_4 defined in (15) satisfies*

$$\frac{d^2}{dt^2} \mathcal{E}_4(t, \mu_t) + \frac{3}{2t} \frac{d}{dt} \mathcal{E}_4(t, \mu_t) \geq 0$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Lagrangian \mathbf{L} given by (14).

Similarly, if we set $m = -\frac{1}{2}$ and $C = 1$ in Corollary 2.5, then $c_1(t) = -2\sqrt{t}$ and $c_2(t) = \frac{1}{2t}$. In this case, the time dependent metric g satisfies $g = \sqrt{t} \bar{g}$, where \bar{g} is a solution of the Ricci flow $\dot{\bar{g}} = -2\bar{\text{Rc}}$. The Lagrangian \mathbf{L} is given by

$$(16) \quad \mathbf{L}(t, x, v) = \frac{\sqrt{t}}{2} \left(\sum_{i,j} \bar{g}_{ij}(t, x) v_i v_j + \bar{R}(t, x) \right).$$

Let $\overline{\text{vol}}$ be the Riemannian volume of the metric \bar{g} . Let \mathcal{E}_5 be the following functional

$$(17) \quad \mathcal{E}_5(t, \mu) = \int_M (\log(\rho(t, x)) - t^{-1/2} u(t, x)) d\mu(x) + \frac{n}{2} \log t,$$

where $\mu = \rho(t, \cdot) \overline{\text{vol}}$.

The following result on \mathcal{E}_5 is also a consequence of Corollary 2.5. This is another result in [20] which leads to the monotonicity of Feldman-Ilmanen-Ni's forward reduced volume [16].

Corollary 2.8. [20, Proposition 20] *The functional \mathcal{E}_5 defined in (17) satisfies*

$$\frac{d^2}{dt^2} \mathcal{E}_5(t, \mu_t) + \frac{3}{2t} \frac{d}{dt} \mathcal{E}_5(t, \mu_t) \geq 0$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Lagrangian \mathbf{L} given by (16).

If we set $C = 1$ and $b \equiv 0$ in Corollary 2.6, then g is a solution of the Ricci flow $\dot{g} = -2\text{Rc}$. The Lagrangian \mathbf{L} is given by

$$(18) \quad \mathbf{L}(t, x, v) = \frac{1}{2} \left(\sum_{i,j} g_{ij}(t, x) v_i v_j + R(t, x) \right).$$

Let \mathcal{E}_6 be the following functional

$$(19) \quad \mathcal{E}_6(t, \mu) = \int_M (\log(\rho(t, x)) - u(t, x)) d\mu(x),$$

where $\mu = \rho(t, \cdot)$ vol. The following result on \mathcal{E}_6 , which is a consequence of Corollary 2.6, appeared in [24, 20].

Corollary 2.9. [20, Corollary 4] *The functional \mathcal{E}_6 defined in (19) satisfies*

$$\frac{d^2}{dt^2} \mathcal{E}_6(t, \mu_t) \geq 0$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Lagrangian \mathbf{L} given by (18).

We also obtain some similar results for functionals given by (6) with $\mathcal{F}(r) = r^q$, where q is a constant. Using the above mentioned approach, we prove the following result which holds for all fibrewise strictly convex Hamiltonians \mathbf{H} defined above.

Theorem 2.10. *Assume that $\mathcal{F}(r) = r^q$. Then the functional \mathbf{F} defined in (6) satisfies the following inequality*

$$\begin{aligned} & \frac{d^2}{dt^2} \mathbf{F}(t, \mu_t) + (qb_1(t) + b_2(t)) \frac{d}{dt} \mathbf{F}(t, \mu_t) + \frac{q b_1(t)^2}{4} + \frac{n b_2(t)^2}{4} \\ & \geq \int_M q (r_{\mathbf{x}}^t)^q \left[\mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - b_2(t) \frac{d}{dt} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \right. \\ & \quad \left. - \frac{d^2}{dt^2} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \right] d\mu_0(\mathbf{x}) \end{aligned}$$

for each smooth displacement interpolation μ_t corresponding to the cost (1).

In the Riemannian case, the convexity of the functional (6) with $\mathcal{F}(r) = r^q$ along displacement interpolations is equivalent to the, so called, curvature-dimension condition. This result, which appeared in

[29], can be obtained as a consequence of Theorem 2.10. More precisely, let \mathcal{E}_7 be the functional defined by

$$(20) \quad \mathcal{E}_7(\mu) = - \int_M \rho(x)^{-1/N} d\mu(x),$$

where $\mu = \rho \text{ vol}$.

Corollary 2.11. [29, Corollary 1.6] *The Ricci curvature Rc of the Riemannian metric g is non-negative and the dimension of the manifold M is $\leq N$ if and only if*

$$(21) \quad \frac{d^2}{dt^2} \mathcal{E}_7(\mu_t) \geq 0$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Lagrangian defined by (3).

Next, we specialize Theorem 2.10 to the case of the Ricci flow. Let \mathbf{H} be the Hamiltonian defined in (11) with the metric g satisfying (12). Let \mathcal{E}_8 be the functional defined by

$$\mathcal{E}_8(t, \mu) = \int_M \rho(t, x)^q d\mu(x)$$

where $\mu = \rho(t, \cdot) \mathbf{m}_t$, $\mathbf{m}_t = e^{-k(t)u(t, \cdot)} \text{vol}$, and k is a functional parameter. In this case, Theorem 2.10 gives to the following result.

Corollary 2.12. *If the functions c_1, c_2, b, U satisfy*

$$c_1 k = -2, \quad \ddot{k} = -b_2 \dot{k}, \quad 2\dot{k} = c_2 k - b_2 k, \quad U = -\frac{c_1^2}{8} R,$$

then

$$\begin{aligned} & \frac{d^2}{dt^2} \mathcal{E}_8(t, \mu_t) + (q b_1(t) + b_2(t)) \frac{d}{dt} \mathcal{E}_8(t, \mu_t) \\ & + q \left(\frac{n \dot{c}_2(t)}{2} + \frac{n c_2(t)^2}{4} + \frac{q b_1(t)^2}{4} + \frac{n b_2(t)^2}{4} \right) \mathcal{E}_8(t, \mu_t) \geq 0 \end{aligned}$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Hamiltonian \mathbf{H} given by (11) and (12).

We remark that, unlike Corollary 2.4, there are two functional parameters, say k and b_1 , which are free in Corollary 2.12. Finally, if we specialize to the case $k(t) = C_1 t^m$ and $b_1(t) = C_2 t^{-1}$, then we obtain the following corollaries.

Corollary 2.13. *If $m \neq 0$, $k(t) = C_1 t^m$, $b_1(t) = C_2 t^{-1}$, $c_1(t) = -\frac{2}{C_1 t^m}$, $c_2(t) = \frac{m+1}{t}$, and $U(t, x) = -\frac{c_1^2}{8} R(t, x)$, then*

$$\begin{aligned} & \frac{d^2}{dt^2} \mathcal{E}_8(t, \mu_t) + \frac{qC_2 - m + 1}{t} \frac{d}{dt} \mathcal{E}_8(t, \mu_t) \\ & + \frac{q(2nm(m-1) + qC_2^2)}{4t^2} \mathcal{E}_8(t, \mu_t) \geq 0 \end{aligned}$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Hamiltonian \mathbf{H} given by (11) and (12).

Corollary 2.14. *If $k \equiv C_1 \neq 0$, $b_1(t) = C_2 t^{-1}$, $c_1 \equiv -\frac{2}{C_1}$, $c_2 = b_2 \equiv \frac{C_3}{t}$, and $U(t, x) = -\frac{c_1^2}{8} R(t, x)$, then*

$$\begin{aligned} & \frac{d^2}{dt^2} \mathcal{E}_8(t, \mu_t) + \left(\frac{qC_2 + C_3}{t} \right) \frac{d}{dt} \mathcal{E}_8(t, \mu_t) \\ & + q \left(\frac{qC_2^2 + 2nC_3^2 - 2nC_3}{4t^2} \right) \mathcal{E}_8(t, \mu_t) \geq 0 \end{aligned}$$

for each smooth displacement interpolation μ_t corresponding to the cost (1) with Hamiltonian \mathbf{H} given by (11) and (12).

3. CURVATURE OF HAMILTONIAN SYSTEMS

In this section, we give a brief discussion on the curvature of a Hamiltonian system introduced in [2]. For a more detail discussion, see for instance [1, 19].

Let T^*M be the cotangent bundle of a n -dimensional manifold M . Let (x_1, \dots, x_n) and $(x_1, \dots, x_n, p_1, \dots, p_n)$ denote systems of local coordinates on M and T^*M , respectively. Let $\omega := \sum_{i=1}^n dp_i \wedge dx_i$ be the canonical symplectic form. Let $\mathbf{H} : \mathbb{R} \times T^*M \rightarrow \mathbb{R}$ be a smooth function, called Hamiltonian, and let $\vec{\mathbf{H}}$ be the corresponding Hamiltonian vector field defined by $\omega(\vec{\mathbf{H}}, \cdot) = -d\mathbf{H}(\cdot)$, where d denotes the exterior differential on T^*M . In local coordinates, $\vec{\mathbf{H}}$ is given by

$$\dot{x}_i = \mathbf{H}_{p_i}, \quad \dot{p}_i = -\mathbf{H}_{x_i}.$$

Recall that the dots here and throughout this paper denotes the derivative with respect to time variables t , s , or τ . We assume that the (possibly time-dependent) vector field $\vec{\mathbf{H}}$ is complete and the corresponding (possibly time-dependent) flow is denoted by $\Phi_{t,s}$ (i.e. $\frac{d}{dt} \Phi_{t,s} = \vec{\mathbf{H}}(\Phi_{t,s})$ and $\Phi_{s,s}$ is the identity map).

Let $\pi : T^*M \rightarrow M$ be the projection to the base M and let \mathcal{V} be the vertical bundle defined as the kernel of the map $d\pi$. Let (\mathbf{x}, \mathbf{p}) be

a point in T^*M and let $J_{(\mathbf{x}, \mathbf{p})}^{t,s}$ be the family of subspaces in $T_{(\mathbf{x}, \mathbf{p})}T^*M$, called Jacobi curve, defined by

$$J_{(\mathbf{x}, \mathbf{p})}^{t,s} = d\Phi_{t,s}^{-1}(\mathcal{V}_{\Phi_{t,s}(\mathbf{x}, \mathbf{p})}).$$

Assume that the Hamiltonian \mathbf{H} is fibrewise strictly convex (i.e. $p \mapsto \mathbf{H}(t, x, p)$ is strictly convex for each time t and each point x). Then the family of bilinear forms $\langle \cdot, \cdot \rangle^{\tau, s}$ defined by

$$\langle v, v \rangle^{\tau, s} := \omega \left(\frac{d}{dt} e^{t,s}, e^{t,s} \right) \Big|_{t=\tau}$$

is an inner product defined on the subspace $J_{(\mathbf{x}, \mathbf{p})}^{t,s}$, where $e^{t,s}$ is contained in $J^{t,s}$ for each time t in a neighborhood of τ and $e^{\tau, s} = v$.

Proposition 3.1. *Assume that the Hamiltonian \mathbf{H} is fibrewise strictly convex. Then there exists a family of bases $e_1^{t,s}, \dots, e_n^{t,s}$ contained in $J^{t,s}$ orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle^{t,s}$ such that $\dot{e}_i^{t,s}$ is contained in $J^{t,s}$. Moreover, if $\bar{e}_1^{t,s}, \dots, \bar{e}_n^{t,s}$ is another such family, then there is an orthonormal matrix O such that*

$$\bar{e}_i^{t,s} = \sum_{j=1}^n O_{ij} e_j^{t,s}.$$

For the proof of the above proposition, see [19, Proposition 3.2]. We define $f_i^{t,s} = \dot{e}_i^{t,s}$ and call the frame

$$(e_1^{t,s}, \dots, e_n^{t,s}, f_1^{t,s}, \dots, f_n^{t,s})^T$$

a canonical frame of $J^{t,s}$. Recall that there are two time parameters t and s for the canonical frame $e_i^{t,s}$ and $f_i^{t,s}$. The dots in $\dot{e}_i^{t,s}$ and $\dot{f}_i^{t,s}$ always denote the derivative with respect to the first time parameter, which is t in this case. Let $\bar{E}^{t,s} = (\bar{e}_1^{t,s}, \dots, \bar{e}_n^{t,s})^T$ be a family of orthonormal bases of $J^{t,s}$. The following gives a procedure for obtaining a canonical frame. See [19, Lemma 3.4] for a proof.

Proposition 3.2. *Let $\Omega^{t,s}$ be the matrix defined by $\Omega_{ij}^{t,s} = \omega(\dot{\bar{e}}_i^{t,s}, \dot{\bar{e}}_j^{t,s})$ and let $O^{t,s}$ be a solution of the equation*

$$\dot{O}^{t,s} = \frac{1}{2} O^{t,s} \Omega^{t,s}.$$

Then $O^{t,s} \bar{E}^{t,s}$ defines a canonical frame of $J^{t,s}$.

If $(e_1^{t,s}, \dots, e_n^{t,s}, f_1^{t,s}, \dots, f_n^{t,s})^T$ is a canonical frame of the Jacobi curve $J_{(\mathbf{x}, \mathbf{p})}^{t,s}$, then the operators $\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^{t,s} : J_{(\mathbf{x}, \mathbf{p})}^{t,s} \rightarrow J_{(\mathbf{x}, \mathbf{p})}^{t,s}$ defined by

$$\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^{t,s} e_i^{t,s} = -\dot{e}_i^{t,s}$$

are the curvature operators of the Jacobi curve $J^{t,s}$. The operator $\mathbf{R}_{(\mathbf{x},\mathbf{p})}^t := \mathbf{R}_{(\mathbf{x},\mathbf{p})}^{t,t} : \mathcal{V}_{(\mathbf{x},\mathbf{p})} \rightarrow \mathcal{V}_{(\mathbf{x},\mathbf{p})}$ is the curvature operator of the Hamiltonian system $\vec{\mathbf{H}}$.

The operators $\mathbf{R}^{t,s}$ are symmetric with respect to the inner product $\langle \cdot, \cdot \rangle^{t,s}$. It also follows from the definition of \mathbf{R} that the following holds

$$\mathbf{R}_{\Phi_{\tau,s}(\mathbf{x},\mathbf{p})}^{t,\tau} = d\Phi_{\tau,s} \mathbf{R}_{(\mathbf{x},\mathbf{p})}^{t,s} (d\Phi_{\tau,s})^{-1}.$$

In particular, the following holds

$$(22) \quad \mathbf{R}_{\Phi_{t,s}(\mathbf{x},\mathbf{p})}^t = d\Phi_{t,s} \mathbf{R}_{(\mathbf{x},\mathbf{p})}^{t,s} (d\Phi_{t,s})^{-1}$$

and it shows that the curvature $\mathbf{R}^{t,s}$ of the Jacobi curve $J^{t,s}$ is completely determined by the curvature \mathbf{R}^t of the Hamiltonian system $\vec{\mathbf{H}}$.

Next, we discuss how to compute the curvature operators. Let V_1^t, \dots, V_n^t be a family of (local) vector fields contained in the vertical bundle \mathcal{V} which is orthonormal with respect to the Riemannian metric on \mathcal{V} defined by

$$v \mapsto \sum_{i,j} \mathbf{H}_{p_i p_j}(t, x, p) v_i v_j.$$

A computation using local coordinates shows that this Riemannian metric is also given by $v \mapsto \omega([\vec{\mathbf{H}}, V], V)$, where V is any extension of v to a vector field contained in \mathcal{V} .

Let $\vec{e}_i^{t,s}$ be defined by

$$\vec{e}_i^{t,s} = (\Phi_{t,s})^* V_i^t.$$

It follows that

$$\begin{aligned} \langle \vec{e}_i^{t,s}, \vec{e}_j^{t,s} \rangle_{(\mathbf{x},\mathbf{p})}^{t,s} &= \omega_{(\mathbf{x},\mathbf{p})}(\vec{e}_i^{t,s}, \vec{e}_j^{t,s}) \\ &= \omega_{\Phi_{t,s}(\mathbf{x},\mathbf{p})}([\vec{\mathbf{H}}, V_i^t], V_j^t) \\ &= \delta_{ij}. \end{aligned}$$

Therefore, we can apply Proposition 3.2 and obtain the following result.

Proposition 3.3. *Let $[\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t]$ be the matrix with ij^{th} entry defined by $\langle \mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t V_i^t, V_j^t \rangle$. Then it satisfies*

$$\begin{aligned} [\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t] \bar{E}^{t,t} &= -\frac{1}{4}(\Omega^{t,t})^2 \bar{E}^{t,t} - \frac{1}{2} \dot{\Omega}^{t,t} \bar{E}^{t,t} - \Omega^{t,t} \dot{\bar{E}}^{t,t} - \ddot{\bar{E}}^{t,t}, \\ \dot{\bar{e}}_i^{t,t} &= [\vec{\mathbf{H}}, V_i^t] + \dot{V}_i^t, \\ \ddot{\bar{e}}_i^{t,t} &= [\vec{\mathbf{H}}, [\vec{\mathbf{H}}, V_i^t]] + 2[\vec{\mathbf{H}}, \dot{V}_i^t] + [\vec{\mathbf{H}}, V_i^t] + \ddot{V}_i^t, \\ \Omega^{t,t} &= \omega(\dot{\bar{e}}_i^{t,t}, \dot{\bar{e}}_j^{t,t}), \\ \dot{\Omega}^{t,t} &= \omega(\ddot{\bar{e}}_i^{t,t}, \dot{\bar{e}}_j^{t,t}) + \omega(\dot{\bar{e}}_i^{t,t}, \ddot{\bar{e}}_j^{t,t}), \end{aligned}$$

where $\bar{E}^{t,t} = (\bar{e}_1^{t,t}, \dots, \bar{e}_n^{t,t})^T$.

Proof. Let $O^{t,s}$ be as in Proposition 3.2 with initial condition $O^{s,s} = I$. Then we have

$$\frac{d}{dt} (O^{t,s} \bar{E}^{t,s}) \Big|_{t=s} = \dot{\bar{E}}^{s,s} + \frac{1}{2} \Omega^{s,s} \bar{E}^{s,s}$$

and

$$\begin{aligned} [\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^s] \bar{E}^{s,s} &= -\frac{d^2}{dt^2} (O^{t,s} \bar{E}^{t,s}) \Big|_{t=s} \\ &= -\frac{d}{dt} \left(\frac{1}{2} O^{t,s} \Omega^{t,s} \bar{E}^{t,s} + O^{t,s} \dot{\bar{E}}^{t,s} \right) \Big|_{t=s} \\ &= -\frac{1}{4} (\Omega^{s,s})^2 \bar{E}^{s,s} - \frac{1}{2} \dot{\Omega}^{s,s} \bar{E}^{s,s} - \Omega^{s,s} \dot{\bar{E}}^{s,s} - \ddot{\bar{E}}^{s,s}. \end{aligned}$$

□

4. OPTIMAL TRANSPORTATION AND HAMILTON-JACOBI EQUATION

In this section, we first recall several known facts about solutions of the Hamilton-Jacobi equation and use them to define the smooth displacement interpolations. Then we define a version of Hessian corresponding to a Hamiltonian. We show that the Hessian of a smooth solution to the Hamilton-Jacobi equation satisfies a matrix Riccati equation. To see the importance of this equation, we show that a Bochner type formula for Hamiltonian system holds under extra assumptions.

First, we recall the following result which is an immediate consequence of the method of characteristics. A proof can be found, for instance, in [10, 14, 4]. (Recall that $\Phi_{t,s}$ is the flow of the Hamiltonian vector field $\vec{\mathbf{H}}$ and $\Phi_t = \Phi_{t,0}$).

Proposition 4.1. *Let u be a smooth solution of the equation*

$$(23) \quad \dot{u} + \mathbf{H}(t, x, du_x) = 0$$

defined on $[0, T] \times M$ with initial condition $u(0, \cdot) = \mathbf{f}(\cdot)$. Then u satisfies the following relation

$$(24) \quad \Phi_t(x, d\mathbf{f}) = (\varphi_t(x), du(t, \cdot)),$$

where $\varphi_t(x) = \pi(\Phi_t(x, d\mathbf{f}))$ and t is in $[0, T]$. Moreover, the path $t \mapsto \varphi_t(x)$ is the unique minimizer of (1) between the endpoints x and $\varphi_T(x)$.

In fact, one can use the relation (24) to give the following short time existence of solution to (23) for any given smooth initial data \mathbf{f} . A detail discussion can be found, for instance, in [4].

Proposition 4.2. *Assume that the manifold M is closed. For any smooth initial data \mathbf{f} , there is a time $T > 0$ such that the equation (23) admits a smooth solution on $[0, T] \times M$ with initial condition $u(0, \cdot) = \mathbf{f}(\cdot)$.*

We define smooth displacement interpolations of Problem 1.1 as any one parameter family of measures defined by (5) where \mathbf{f} is any smooth function such that (23) admits a smooth solution on $[0, T] \times M$ with initial condition $u(0, \cdot) = \mathbf{f}(\cdot)$. The terminology is justified by the following proposition. The proof is a simple application of Proposition 4.1 and the Kantorovich duality (see [33, 34]). The proof can be found in [18].

Proposition 4.3. *Let μ_t be a smooth displacement interpolation corresponding to the cost (1) defined by the function \mathbf{f} . Then the map*

$$\varphi_T(x) := \pi(\Phi_T(d\mathbf{f}_x))$$

is a solution of Problem 1.1 between any measure μ and $(\varphi_T)_\mu$.*

Next, we introduce Hessian of the function \mathbf{f} with respect to the Hamiltonian system $\vec{\mathbf{H}}$. Let us consider the map $x \mapsto d\mathbf{f}_x$. The differential of this map sends the tangent space $T_x M$ to a Lagrangian subspace of the symplectic vector space $T_{(x, d\mathbf{f})} T^* M$. This is the graph of a linear map from the horizontal space

$$\mathcal{H}_{(x, d\mathbf{f})}^t = \text{span}\{f_1^{t,t}, \dots, f_n^{t,t}\}$$

to the vertical space $\mathcal{V}_{(x, d\mathbf{f})}$. The negative of this linear map is the Hessian of \mathbf{f} with respect to the Hamiltonian system $\vec{\mathbf{H}}$ at (t, x) and it is denoted by $\nabla_{\vec{\mathbf{H}}}^2 \mathbf{f}(t, x)$. The trace with respect to the canonical frame $E^{t,t} = (e_1^{t,t}, \dots, e_n^{t,t})^T$ and $F^{t,t} = (f_1^{t,t}, \dots, f_n^{t,t})^T$ at $(x, d\mathbf{f})$ is denoted by $\Delta_{\vec{\mathbf{H}}} \mathbf{f}(t, x)$.

Let $E^{t,s} = (e_1^{t,s}, \dots, e_n^{t,s})^T$ and $F^{t,s} = (f_1^{t,s}, \dots, f_n^{t,s})^T$ be a canonical frame of the Jacobi curve $J_{(\mathbf{x}, d\mathbf{f})}^{t,s} = d\Phi_{t,s}^{-1}(\mathcal{V}_{\Phi_{t,s}(\mathbf{x}, d\mathbf{f})})$. Note that the following relation holds

$$d\Phi_\tau(J_{(\mathbf{x}, d\mathbf{f})}^{t,0}) = J_{\Phi_\tau(\mathbf{x}, d\mathbf{f})}^{t,\tau} = J_{(\varphi_\tau(\mathbf{x}), du(t, \cdot))}^{t,\tau}.$$

It follows that $\tilde{E}^{t,\tau} = (\tilde{e}_1^{t,\tau}, \dots, \tilde{e}_n^{t,\tau})^T$ and $\tilde{F}^{t,\tau} = (\tilde{f}_1^{t,\tau}, \dots, \tilde{f}_n^{t,\tau})^T$ defined by

$$\begin{aligned}\tilde{E}^{t,\tau} &:= d\Phi_\tau(E^{t,0}) = (d\Phi_\tau(e_1^{t,0}), \dots, d\Phi_\tau(e_n^{t,0}))^T \\ \tilde{F}^{t,\tau} &:= d\Phi_\tau(F^{t,0}) = (d\Phi_\tau(f_1^{t,0}), \dots, d\Phi_\tau(f_n^{t,0}))^T\end{aligned}$$

is a canonical frame of the Jacobi curve $J_{\Phi_\tau(\mathbf{x}, d\mathbf{f})}^{t,\tau}$.

Let $A(t)$ and $B(t)$ be the matrices defined by

$$(25) \quad d(d\mathbf{f})_{\mathbf{x}}(d\pi(f_i^{0,0})) = \sum_{j=1}^n (A_{ij}(t)e_j^{t,0} + B_{ij}(t)f_j^{t,0}).$$

where $d(d\mathbf{f})_{\mathbf{x}}$ is the differential of the map $x \mapsto (d\mathbf{f})_x$ at $x = \mathbf{x}$.

By applying $d\Phi_t$ on both sides of (25), we obtain the following relation

$$(26) \quad d\Phi_t(d(d\mathbf{f})_{\mathbf{x}}(d\pi(f_i^{0,0}))) = \sum_{j=1}^n (A_{ij}(t)\tilde{e}_j^{t,t} + B_{ij}(t)\tilde{f}_j^{t,t}).$$

It follows from this equation and (24) that

$$d(du(t, \cdot))_{\varphi_t(\mathbf{x})}(d\varphi_t(d\pi(f_i^{0,0}))) = \sum_{j=1}^n (A_{ij}(t)\tilde{e}_j^{t,t} + B_{ij}(t)\tilde{f}_j^{t,t}).$$

If we apply (26) to this equation, then we obtain

$$\sum_{j=1}^n B_{ij}(t)d(du(t, \cdot))_{\varphi_t(\mathbf{x})}(d\pi(\tilde{f}_j^{t,t})) = \sum_{j=1}^n (A_{ij}(t)\tilde{e}_j^{t,t} + B_{ij}(t)\tilde{f}_j^{t,t}).$$

Therefore, the matrix representation $S(t)$ of the Hessian $\nabla_{\mathbf{H}}^2 u(t, \varphi_t(\mathbf{x}))$ (as a map from the horizontal space $\mathcal{H}_{(\varphi_t(\mathbf{x}), du(t, \cdot))}^t$ to the vertical space $\mathcal{V}_{(\varphi_t(\mathbf{x}), du(t, \cdot))}$) with respect to the basis $\tilde{E}^{t,t}$ and $\tilde{F}^{t,t}$ is given by

$$(27) \quad S(t) = -B(t)^{-1}A(t).$$

By differentiating (25) and using the relation (22), we also obtain

$$\dot{B}(t) = -A(t), \quad \dot{A}(t) = B(t)[\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t],$$

where $[\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t]$ denotes the matrix representation of $\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t$ with respect to the basis $E^{t,t}$.

It follows from these equations and (27) that the following matrix Riccati equation holds

$$(28) \quad \dot{S}(t) + S(t)^2 + [\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t] = 0.$$

The equation (28) can be seen as a Hamiltonian analogue of Bochner formula in Riemannian geometry. For this, we use an argument in [34]. Indeed, assume that the map $f \mapsto \Delta_{\mathbf{H}}f$ is linear. Then we can take the trace of the Riccati equation (28) and obtain the following

$$\begin{aligned} & -\Delta_{\mathbf{H}}\mathbf{H}(t, \varphi_t(x), du) + d(\Delta_{\mathbf{H}}u)(\mathbf{H}_p(\varphi_t(x))) \\ & = \Delta_{\mathbf{H}}\dot{u}(t, \varphi_t(x)) + d(\Delta_{\mathbf{H}}u)(\dot{\varphi}_t(x)) \\ & = \frac{d}{dt}(\Delta_{\mathbf{H}}u(t, \varphi_t(x))) \\ & = -|\nabla_{\mathbf{H}}^2 u(t, \varphi_t(x))|^2 - \mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t). \end{aligned}$$

Here $|\nabla_{\mathbf{H}}^2 u(t, \varphi_t(x))|$ denotes the Hilbert-Schmidt norm of the linear operator $\nabla_{\mathbf{H}}^2 u_t : \mathcal{H}_{\varphi_t(x)}^t \rightarrow \mathcal{V}_{\varphi_t(x)}$ with respect to the Euclidean structure defined by a canonical frame $E^{t,t}$ on the vertical space $\mathcal{V}_{\varphi_t(x)}$ and $F^{t,t}$ on the horizontal space $\mathcal{H}_{\varphi_t(x)}^t$.

By setting $t = 0$, we obtain the following Bochner type formula

$$\Delta_{\mathbf{H}}\mathbf{H}(t, x, d\mathbf{f}) - d(\Delta_{\mathbf{H}}\mathbf{f})(\mathbf{H}_p(x, d\mathbf{f})) = |\nabla_{\mathbf{H}}^2 \mathbf{f}(x)|^2 + \mathbf{tr}(\mathbf{R}_{(\mathbf{x}, d\mathbf{f})}^0).$$

Remark 4.4. One can use the matrix Riccati equation (28) together with comparison results in ([28]) to prove analogues of Laplacian, Hessian, and volume comparison theorems for Hamiltonian systems.

Remark 4.5. By combining the work in [3] with the above argument, one can recover the subriemannian Bochner formula in [5] (at least in the three dimensional case) in a geometric way.

5. NATURAL MECHANICAL HAMILTONIANS

In this section, we discuss how to compute the Hessian and the curvature of the natural mechanical Hamiltonians. Let g be a Riemannian metric, let $g_{ij} = g(\partial_{x_i}, \partial_{x_j})$, and let g^{ij} be the inverse matrix of g_{ij} . Let U be a smooth function on the manifold M . The Hamiltonian given in local coordinates by

$$(29) \quad \mathbf{H}(x, p) = \frac{1}{2} \sum_{i,j} g^{ij}(x) p_i p_j + U(t, x)$$

is called a natural mechanical Hamiltonian. When $U \equiv 0$, the Hamiltonian flow of \mathbf{H} coincides with the geodesic flow of g . The proof of the

following illustrate how Proposition 3.3 can be applied and it is needed in the later sections. For an alternative proof, see [1].

Proposition 5.1. *The Hessian $\nabla_{\mathbf{H}}^2 \mathbf{f}$ and the curvature \mathbf{R} of the Hamiltonian (29) are given by*

$$\nabla_{\mathbf{H}}^2 \mathbf{f}(t, x) = \nabla^2 \mathbf{f}(x)$$

and

$$\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t(V) = Rm_{\mathbf{x}}(\mathbf{p}, V)\mathbf{p} + \nabla^2 U_{\mathbf{x}}(V),$$

respectively. Here Rm and ∇^2 denote the Riemannian curvature tensor and the Hessian, respectively, of the given Riemannian metric g .

Remark 5.2. In Proposition 5.1, we have made several identifications $T_{\mathbf{x}}M \cong T_{\mathbf{x}}^*M \cong \mathcal{V}_{(\mathbf{x}, \mathbf{p})}$. The first identification is done by the Riemannian metric $v \mapsto g(v, \cdot)$ and the second one is by $l \mapsto \frac{d}{dt}(\mathbf{p} + tl) \Big|_{t=0}$. We will make these identifications for the rest of this paper without mentioning.

Proof. Let us fix a geodesic normal coordinates (x_1, \dots, x_n) around a point \mathbf{x} and let $(x_1, \dots, x_n, p_1, \dots, p_n)$ be the corresponding coordinates on the cotangent bundle around the point $(\mathbf{x}, d\mathbf{f})$. The Hamiltonian vector field $\vec{\mathbf{H}}$ is given by

$$\vec{\mathbf{H}} = \sum_{i,j} g^{ij} p_j \partial_{x_i} - \sum_k \left(\frac{1}{2} \sum_{i,j} g_{x_k}^{ij} p_i p_j + U_{x_k} \right) \partial_{p_k}.$$

Let $V_m^t = V_m = \sum_i \sqrt{g_{mi}} \partial_{p_i}$ which is independent of time t . First, we compute the bracket of $\vec{\mathbf{H}}$ and V_m

$$[\vec{\mathbf{H}}, V_m] = \sum_{i,j,l} g^{ij} p_j (\sqrt{g_{ml}})_{x_i} \partial_{p_l} - \sum_i \sqrt{g^{mi}} \partial_{x_i} + \sum_{i,l,k} \sqrt{g_{ml}} (g_{x_k}^{il} p_i) \partial_{p_k}.$$

Using the properties of geodesic normal coordinates, we have $g^{ij} = \delta^{ij}$ and $(\sqrt{g_{ij}})_{x_k} = 0$ at \mathbf{x} . Therefore, the followings hold at $(\mathbf{x}, d\mathbf{f})$

$$[\vec{\mathbf{H}}, V_m] = -\partial_{x_m}, \quad \Omega_{ij}^{0,0} = \omega([\vec{\mathbf{H}}, V_i], [\vec{\mathbf{H}}, V_j]) = 0.$$

It follows that the horizontal space at $(\mathbf{x}, d\mathbf{f})$ is spanned by $\partial_{x_1}, \dots, \partial_{x_n}$. Since

$$d(d\mathbf{f})_{\mathbf{x}}(-\partial_{x_i}) = -\sum_j \mathbf{f}_{x_i x_j} \partial_{p_j} - \partial_{x_i},$$

$\nabla_{\mathbf{H}}^2 \mathbf{f}$ is given by $\mathbf{f}_{x_i x_j} = \nabla^2 \mathbf{f}$ at $(\mathbf{x}, d\mathbf{f})$.

Since $(\sqrt{g}_{lk})_{x_i x_r} = \frac{1}{2}(g_{lk})_{x_i x_r}$ at \mathbf{x} , a computation shows that the following holds at \mathbf{x}

$$\begin{aligned} & [\vec{\mathbf{H}}, [\vec{\mathbf{H}}, V_m]] \\ &= \frac{1}{2} \sum_{i,j,k} p_i p_j \left((g_{mk})_{x_i x_j} + (g_{ij})_{x_k x_m} - 2(g_{im})_{x_k x_j} \right) \partial_{p_k} - \sum_k U_{x_k x_m} \partial_{p_k}. \end{aligned}$$

and

$$\begin{aligned} & \omega([\vec{\mathbf{H}}, [\vec{\mathbf{H}}, V_m]], [\vec{\mathbf{H}}, V_l]) \\ &= -\frac{1}{2} \sum_{i,j} p_i p_j \left((g_{ml})_{x_i x_j} + (g_{ij})_{x_l x_m} - 2(g_{im})_{x_l x_j} \right) + U_{x_l x_m}. \end{aligned}$$

It follows from Proposition 3.3 that

$$\dot{\Omega}_{ml}^{0,0} = \sum_{i,j} p_i p_j \left((g_{im})_{x_l x_j} - (g_{il})_{x_m x_j} \right)$$

at \mathbf{x} (note that since $\vec{H} = \vec{U}$ is contained in \mathcal{V} , $[\vec{H}, V_i] = 0$).

Therefore,

$$\begin{aligned} & [\vec{\mathbf{H}}, [\vec{\mathbf{H}}, V_m]] + \Omega_{ml}^{0,0} [\vec{\mathbf{H}}, V_l] + \frac{1}{2} \dot{\Omega}_{ml}^{0,0} V_l + \frac{1}{4} \Omega_{mn}^{0,0} \Omega_{nl}^{0,0} V_l \\ &= \frac{1}{2} \sum_{i,j,k} p_i p_j \left((g_{mk})_{x_i x_j} + (g_{ij})_{x_k x_m} \right. \\ &\quad \left. - (g_{im})_{x_k x_j} - (g_{ik})_{x_m x_j} \right) \partial_{p_k} - \sum_k U_{x_k x_m} \partial_{p_k}. \end{aligned}$$

Recall that the Riemann curvature tensor Rm satisfies the following equation at \mathbf{x}

$$R_{ijk}^l = \frac{1}{2} \left((g_{ik})_{x_j x_l} - (g_{kj})_{x_i x_l} - (g_{il})_{x_j x_k} + (g_{lj})_{x_i x_k} \right),$$

where R_{ijk}^l is defined by $Rm(\partial_{x_i}, \partial_{x_j})\partial_{x_k} = \sum_l R_{ijk}^l \partial_{x_l}$.

Therefore, by combining all of the above computations with Proposition 3.3, we obtain the following equation at \mathbf{x}

$$\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t(V) = Rm_{\mathbf{x}}(\mathbf{p}, V)\mathbf{p} + \nabla^2 U_{\mathbf{x}}(V).$$

□

6. NATURAL MECHANICAL HAMILTONIANS WITH EVOLVING RIEMANNIAN METRICS

In this section, we consider the case when both the Riemannian metric g and the potential U depend on time. Let

$$(30) \quad \mathbf{H}(t, x, p) = \frac{1}{2} \sum_{i,j} g^{ij}(t, x) p_i p_j + U(t, x)$$

be the Hamiltonian with Hamiltonian vector field

$$\vec{\mathbf{H}} = \sum_{i,j} g^{ij} p_j \partial_{x_i} - \sum_k \left(\frac{1}{2} \sum_{i,j} g_{x_k}^{ij} p_i p_j + U_{x_k} \right) \partial_{p_k}.$$

Theorem 6.1. *The Hessian $\nabla_{\mathbf{H}}^2 \mathbf{f}$ and the curvature \mathbf{R} of the Hamiltonian system $\vec{\mathbf{H}}$ defined by (30) satisfy*

$$\nabla_{\mathbf{H}}^2 \mathbf{f}(t, \mathbf{x}) = \nabla^2 \mathbf{f}(t, \mathbf{x}) + \frac{1}{2} \dot{g}(t, \mathbf{x})$$

and

$$\begin{aligned} \mathbf{tr}(\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t) &= Rc(t, \mathbf{x})(\mathbf{p}, \mathbf{p}) + \Delta U(t, \mathbf{x}) - \langle \nabla(\mathbf{tr}(\dot{g}(t, \mathbf{x}))), \mathbf{p} \rangle \\ &\quad + \mathbf{div}(\dot{g}(t, \mathbf{x}))(\mathbf{p}) - \frac{1}{2} \mathbf{tr}(\ddot{g}(t, \mathbf{x})) + \frac{1}{4} |\dot{g}(t, \mathbf{x})|^2 \end{aligned}$$

where Rc , ∇ , and Δ are the Ricci curvature, the gradient, and the Laplacian, respectively, of the Riemannian metric $g(t, \cdot)$ at time t .

Proof of Theorem 6.1. We apply Proposition 3.3 to the time dependent Hamiltonian (30). Let us fix a geodesic normal coordinate around \mathbf{x} with respect to the Riemannian metric g at time t . A computation shows that

$$[\vec{\mathbf{H}}, \sum_i a_i^t(x) \partial_{p_i}] = \sum_{i,j,l} g^{lj} p_j (a_i^t)_{x_l} \partial_{p_i} - \sum_{j,l} g^{lj} a_j^t \partial_{x_l} + \sum_{j,k,l} g_{x_k}^{lj} p_j a_l^t \partial_{p_k}.$$

Therefore, the following equation holds at (\mathbf{x}, \mathbf{p})

$$(31) \quad [\vec{\mathbf{H}}, \sum_i a_i^t(x) \partial_{p_i}] = \sum_{i,l} p_l (a_i^t)_{x_l} \partial_{p_i} - \sum_l a_l^t \partial_{x_l}.$$

Let $V_i^t = \sum_j \sqrt{g_{ij}} \partial_{p_j}$ and let $\bar{e}_i^{t,s} = (\Phi_{t,s})^* V_i^t$. Then clearly $\bar{e}_i^{t,t} = \partial_{p_i}$ at (\mathbf{x}, \mathbf{p}) . Since $\dot{\sqrt{g}}_{ij} = \frac{1}{2} \dot{g}_{ij}$ at \mathbf{x} , it follows from (31) that

$$\bar{f}_i^{t,t} := \dot{\bar{e}}_i^{t,t} = [\vec{\mathbf{H}}, V_i^t] + \dot{V}_i^t = -\partial_{x_i} + \sum_j \frac{1}{2} \dot{g}_{ij} \partial_{p_j}$$

at (\mathbf{x}, \mathbf{p}) .

It follows that

$$\begin{aligned} d(df)_{\mathbf{x}}(-\partial_{x_i}) &= -\sum_j f_{x_i x_j} \partial_{p_j} - \partial_{x_i} \\ &= -\sum_j \left(f_{x_i x_j} + \frac{1}{2} \dot{g}_{ij} \right) \partial_{p_j} - \partial_{x_i} + \frac{1}{2} \sum_j \dot{g}_{ij} \partial_{p_j} \end{aligned}$$

and so

$$\nabla_{\mathbf{H}}^2 \mathbf{f} = \nabla^2 \mathbf{f} + \frac{1}{2} \dot{g}.$$

Since \dot{g}_{ij} is symmetric, it follows that

$$\Omega_{ij}^{t,t} = \omega(\bar{f}_i^{t,t}, \bar{f}_j^{t,t}) = 0.$$

Clearly, we have

$$\dot{V}_i^t = \sum_k \dot{\sqrt{g}}_{ik} \partial_{p_k} = \frac{1}{2} \sum_k \dot{g}_{ik} \partial_{p_k}$$

at \mathbf{x} . It follows from (31) that

$$[\vec{\mathbf{H}}, \dot{V}_i^t] = \frac{1}{2} \sum_{l,m} p_l (\dot{g}_{im})_{x_l} \partial_{p_m} - \sum_m \frac{1}{2} \dot{g}_{im} \partial_{x_m}$$

at \mathbf{x} .

By a computation together with the facts that $\dot{g}_{ij} = -\dot{g}^{ij}$ and $(\dot{g}_{ij})_{x_k} = -(\dot{g}^{ij})_{x_k}$, we see that

$$[\vec{\mathbf{H}}_t, V_i^t] = \sum_l \dot{g}_{il} \partial_{x_l} - \sum_{l,k} (\dot{g}_{il})_{x_k} p_l \partial_{p_k}$$

at \mathbf{x} .

Since $\ddot{\sqrt{g}}_{ik} = \frac{1}{2} \ddot{g}_{ik} - \sum_j \frac{1}{4} \dot{g}_{ij} \dot{g}_{jk}$, we have

$$\ddot{V}_i^t = \sum_k \ddot{\sqrt{g}}_{ik} \partial_{p_k} = \sum_k \left(\frac{1}{2} \ddot{g}_{ik} - \frac{1}{4} \sum_j \dot{g}_{ij} \dot{g}_{jk} \right) \partial_{p_k}$$

at \mathbf{x} .

By combining all of the above computations, we obtain

$$\begin{aligned} \dot{\bar{f}}_i^{t,t} &= [\vec{\mathbf{H}}, [\vec{\mathbf{H}}, V_i^t]] + 2[\vec{\mathbf{H}}, \dot{V}_i^t] + \ddot{V}_i^t + [\vec{\mathbf{H}}_t, V_i^t] \\ &= [\vec{\mathbf{H}}, [\vec{\mathbf{H}}, V_i^t]] + \sum_k \left(\sum_l p_l (\dot{g}_{ik})_{x_l} - \sum_l (\dot{g}_{li})_{x_k} p_l \right. \\ &\quad \left. + \frac{1}{2} \ddot{g}_{ik} - \frac{1}{4} \sum_j \dot{g}_{ij} \dot{g}_{jk} \right) \partial_{p_k}. \end{aligned}$$

From the proof of Proposition 5.1, the vector field $[\vec{\mathbf{H}}, [\vec{\mathbf{H}}, V_i^t]]$ is contained in the vertical space $\mathcal{V}_{(\mathbf{x}, \mathbf{p})}$ at (\mathbf{x}, \mathbf{p}) . Therefore, it follows from the above computations that

$$\begin{aligned} \omega(\ddot{\vec{e}}_i^{t,t}, \dot{\vec{e}}_j^{t,t}) &= \omega([\vec{\mathbf{H}}, [\vec{\mathbf{H}}, V_i^t]], [\vec{\mathbf{H}}, V_j^t]) - \sum_l p_l (\dot{g}_{ij})_{x_l} \\ &\quad - \frac{1}{2} \ddot{g}_{ij} + \frac{1}{4} \sum_k \dot{g}_{ik} \dot{g}_{kj} + \sum_l (\dot{g}_{li})_{x_j} p_l. \end{aligned}$$

Hence,

$$\begin{aligned} \dot{\Omega}_{ij}^{t,t} &= \omega(\ddot{\vec{e}}_i^{t,t}, \dot{\vec{e}}_j^{t,t}) - \omega(\ddot{\vec{e}}_j^{t,t}, \dot{\vec{e}}_i^{t,t}) \\ &= \omega([\vec{\mathbf{H}}, [\vec{\mathbf{H}}, V_i^t]], [\vec{\mathbf{H}}, V_j^t]) - \omega([\vec{\mathbf{H}}, [\vec{\mathbf{H}}, V_j^t]], [\vec{\mathbf{H}}, V_i^t]) \\ &\quad + \sum_l (\dot{g}_{li})_{x_j} p_l - \sum_l (\dot{g}_{lj})_{x_i} p_l. \end{aligned}$$

Finally, from the proof of Proposition 5.1 again, we obtain the following equation at (\mathbf{x}, \mathbf{p})

$$\begin{aligned} -\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t &= \frac{1}{2} \dot{\Omega}^{t,t} \tilde{E}^{t,t} + \dot{\tilde{F}}^{t,t} \\ &= \sum_k \left(- \sum_{i,r} R_{klr}^i p_i p_r - U_{x_l x_k} + \sum_l (\dot{g}_{ik})_{x_l} p_l - \frac{1}{2} \sum_l (\dot{g}_{li})_{x_k} p_l \right. \\ &\quad \left. - \frac{1}{2} \sum_l (\dot{g}_{lk})_{x_i} p_l + \frac{1}{2} \ddot{g}_{ik} - \frac{1}{4} \sum_j \dot{g}_{ij} \dot{g}_{jk} \right) \partial_{p_k} \end{aligned}$$

and

$$\begin{aligned} \text{tr}(\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t) &= \sum_{l,r,i} R_{llr}^i p_i p_r + \sum_l U_{x_l x_l} \\ &\quad - \left(\sum_{i,l} (\dot{g}_{ii})_{x_l} p_l - \sum_{i,l} (\dot{g}_{li})_{x_i} p_l + \frac{1}{2} \sum_i \ddot{g}_{ii} - \frac{1}{4} \sum_{i,j} \dot{g}_{ij} \dot{g}_{ji} \right). \end{aligned}$$

□

Finally, we look at the following special case.

Theorem 6.2. *Let us fix two smooth functions $c_1, c_2 : \mathbb{R} \rightarrow \mathbb{R}$ of time t . Let g be a smooth solution of the following equation*

$$(32) \quad \dot{g} = c_1 R c + c_2 g.$$

Then

$$\begin{aligned} \mathbf{tr}(\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t) &= Rc(t, \mathbf{x})(\mathbf{p}, \mathbf{p}) + \Delta U(t, \mathbf{x}) - \frac{1}{2}c_1(t) \langle \nabla R(t, x), \mathbf{p} \rangle \\ &\quad - \frac{\dot{c}_1(t)}{2}R(t, \mathbf{x}) + \frac{c_1(t)^2}{4}|Rc(t, \mathbf{x})|^2 + \frac{c_1(t)^2}{4}\Delta R(t, \mathbf{x}) - \frac{n\dot{c}_2(t)}{2} - \frac{nc_2(t)^2}{4}, \end{aligned}$$

where R is the scalar curvature of the metric g at time t .

Proof. By taking the trace of (32), we obtain

$$\mathbf{tr}(\dot{g}) = c_1 R + c_2 n.$$

On the other hand, if we take the covariant derivative of (32), then

$$\nabla_{v_i} \dot{g}(\mathbf{p}, v_i) = c_1 \nabla_{v_i} Rc(\mathbf{p}, v_i).$$

Therefore, if we sum over i and use the twice contracted Bianchi identity [11, (1.19)], then

$$\mathbf{div}(\dot{g}) = \frac{1}{2}c_1 dR.$$

If we take the divergence again, then we obtain

$$\mathbf{div}(\mathbf{div}(\dot{g})) = \frac{1}{2}c_1 \Delta R.$$

By using (32) again, we obtain

$$\langle \dot{g}, Rc \rangle = \langle c_1 Rc + c_2 g, Rc \rangle = c_1 |Rc|^2 + c_2 R.$$

Therefore, by combining the above calculations with [11, Lemma 2.7], we have

$$\begin{aligned} \dot{R} &= -\Delta(\mathbf{tr}(\dot{g})) + \mathbf{div}(\mathbf{div}(\dot{g})) - \langle \dot{g}, Rc \rangle \\ (33) \quad &= -\frac{c_1}{2}\Delta R - c_1 |Rc|^2 - c_2 R. \end{aligned}$$

A simple calculation also gives

$$|\dot{g}|^2 = |c_1 Rc + c_2 g|^2 = c_1^2 |Rc|^2 + 2c_1 c_2 R + n c_2^2.$$

Therefore, the above calculations reduce Theorem 6.1 to

$$\begin{aligned} \mathbf{tr}(\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t) &= Rc_{\mathbf{x}}(\mathbf{p}, \mathbf{p}) + \Delta U(\mathbf{x}) - \frac{1}{2}c_1 \langle \nabla R, \mathbf{p} \rangle \\ &\quad - \frac{1}{2} \left(\dot{c}_1 R + c_1 \dot{R} + n\dot{c}_2 \right) - \frac{1}{4}(c_1^2 |Rc|^2 + 2c_1 c_2 R + n c_2^2) \\ &= Rc_{\mathbf{x}}(\mathbf{p}, \mathbf{p}) + \Delta U(\mathbf{x}) - \frac{1}{2}c_1 \langle \nabla R, \mathbf{p} \rangle \\ &\quad - \frac{\dot{c}_1}{2}R - \frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4} + \frac{c_1^2}{4}|Rc|^2 + \frac{c_1^2}{4}\Delta R. \end{aligned}$$

□

7. PROOF OF THEOREM 2.2 AND COROLLARY 2.3

In this section, we give the proof of Theorem 2.2 and show that Corollary 2.3 can be seen as a consequence. Before this, we need to introduce the volume distortion factor v in the statement of the theorem. It is defined as follows

$$v(t, x, p) = \log \left((\pi^* \mathbf{m}_t)_{(x,p)}(f_1^{t,t}, \dots, f_n^{t,t}) \right).$$

Clearly, the definition is independent of the choice of a canonical frame due to Proposition 3.1.

Let $\mathcal{F} : \mathbb{R} \rightarrow \mathbb{R}$ and recall that the functional \mathbf{F} is defined by

$$\mathbf{F}(t, \mu) = \int_M \mathcal{F}(\rho(t, x)) d\mu(x),$$

where $\mu = \rho(t, \cdot) \mathbf{m}_t$.

Let $\mathbf{f} : M \rightarrow \mathbb{R}$ be a smooth function which defines the smooth displacement interpolation μ_t . Recall that this means

$$\mu_t = (\varphi_t)_* \mu$$

where $\varphi_t(x) = \pi(\Phi_t(d\mathbf{f}_x))$. Let u be the solution of the equation (23) with initial condition $u(0, \cdot) = \mathbf{f}(\cdot)$. First we have the following lemma.

Lemma 7.1. *The functional \mathbf{F} satisfies*

$$\begin{aligned} & \frac{d}{dt} \mathbf{F}(t, \mu_t) \\ &= - \int_M \mathcal{F}'(r_{\mathbf{x}}^t) r_{\mathbf{x}}^t \left(\frac{d}{dt} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) + \Delta_{\mathbf{H}} u(t, \varphi_t(\mathbf{x})) \right) d\mu(\mathbf{x}) \end{aligned}$$

and

$$\begin{aligned} & \frac{d^2}{dt^2} \mathbf{F}(t, \mu_t) \\ &= \int_M \left(\mathcal{F}''(r_{\mathbf{x}}^t) r_{\mathbf{x}}^t + \mathcal{F}'(r_{\mathbf{x}}^t) \right) r_{\mathbf{x}}^t \left(\frac{d}{dt} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) + \Delta_{\mathbf{H}} u(t, \varphi_t(\mathbf{x})) \right)^2 \\ & \quad + \mathcal{F}'(r_{\mathbf{x}}^t) r_{\mathbf{x}}^t \left(|\nabla_{\mathbf{H}}^2 u(t, \varphi_t(x))|^2 + \mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - \frac{d^2}{dt^2} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \right) d\mu(\mathbf{x}). \end{aligned}$$

Proof. Let $E^{t,s} = (e_1^{t,s}, \dots, e_n^{t,s})^T$ be a canonical frame at $(\mathbf{x}, d\mathbf{f})$ and let $\tilde{E}^{t,\tau} = (\tilde{e}_1^{t,\tau}, \dots, \tilde{e}_n^{t,\tau})^T$ be the canonical frame at $\Phi_t(\mathbf{x}, d\mathbf{f})$ defined by

$$\tilde{e}_i^{t,\tau} = d\Phi_\tau(e_i^{t,0}).$$

Let $A(t)$ and $B(t)$ be as in (25). It follows from (26) that

$$d\varphi_t(d\pi(f_i^{0,0})) = \sum_{j=1}^n B_{ij}(t) d\pi(\tilde{f}_j^{t,t}).$$

Let $\rho(t, \cdot)$ be the function defined by $\mu_t := (\varphi_t)_* \mu = \rho(t, \cdot) \mathbf{m}_t$. The two functions $\rho(0, \cdot)$ and $\rho(t, \cdot)$ are related by

$$\rho(0, \mathbf{x}) e^{v(0, \mathbf{x}, d\mathbf{f})} = \rho(t, \varphi_t(\mathbf{x})) \det B(t) e^{v(t, \Phi_t(\mathbf{x}, d\mathbf{f}))}.$$

If we differentiate this with respect to t , then we obtain

$$\dot{r}_{\mathbf{x}}^t = -r_{\mathbf{x}}^t \left(\frac{d}{dt} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) + \mathbf{tr}(S(t)) \right),$$

where $r_{\mathbf{x}}^t = \rho(t, \varphi_t(\mathbf{x}))$ and $S(t) = -B(t)^{-1}A(t)$.

It follows that

$$\begin{aligned} \frac{d}{dt} \mathbf{F}(t, \mu_t) &= \frac{d}{dt} \int_M \mathcal{F}(\rho(t, \varphi_t(\mathbf{x}))) d\mu(\mathbf{x}) \\ &= \int_M \mathcal{F}'(r_{\mathbf{x}}^t) \dot{r}_{\mathbf{x}}^t d\mu(\mathbf{x}) \\ &= - \int_M \mathcal{F}'(r_{\mathbf{x}}^t) r_{\mathbf{x}}^t \left(\frac{d}{dt} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) + \mathbf{tr}(S(t)) \right) d\mu(\mathbf{x}) \end{aligned}$$

and the first statement of the lemma follows.

If we differentiate the above with respect to time t , then we obtain

$$\begin{aligned} \frac{d^2}{dt^2} \mathbf{F}(t, \mu_t) &= \int_M \mathcal{F}''(r_{\mathbf{x}}^t) (\dot{r}_{\mathbf{x}}^t)^2 + \mathcal{F}'(r_{\mathbf{x}}^t) \ddot{r}_{\mathbf{x}}^t d\mu(\mathbf{x}) \\ &= \int_M \left(\mathcal{F}''(r_{\mathbf{x}}^t) r_{\mathbf{x}}^t + \mathcal{F}'(r_{\mathbf{x}}^t) \right) r_{\mathbf{x}}^t \left(\frac{d}{dt} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) + \mathbf{tr}(S(t)) \right)^2 \\ &\quad + \mathcal{F}'(r_{\mathbf{x}}^t) r_{\mathbf{x}}^t \left(|S(t)|^2 + \mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - \frac{d^2}{dt^2} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \right) d\mu(\mathbf{x}) \end{aligned}$$

as claimed. \square

Proof of Theorem 2.2. By Lemma 7.1, we have

$$\begin{aligned} \frac{d^2}{dt^2} \mathbf{F}(t, \mu_t) &\geq \int_M |S(t)|^2 + \mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - \frac{d^2}{dt^2} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) d\mu(\mathbf{x}) \\ &\geq \int_M b(t) \mathbf{tr}(S(t)) - \frac{nb(t)^2}{4} + \mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - \frac{d^2}{dt^2} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) d\mu(\mathbf{x}). \end{aligned}$$

Therefore, by Lemma 7.1 again, we obtain

$$\begin{aligned} & \frac{d^2}{dt^2} \mathbf{F}(t, \mu_t) + b(t) \frac{d}{dt} \mathbf{F}(t, \mu_t) \\ & \geq \int_M \mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - \frac{nb(t)^2}{4} - \frac{d^2}{dt^2} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \\ & \quad - b(t) \frac{d}{dt} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) d\mu(\mathbf{x}). \end{aligned}$$

□

Proof of Corollary 2.3. Since $b \equiv 0$ and $\mathcal{F}(r) = \log r$ in this case, Theorem 2.2 reduces to

$$\frac{d^2}{dt^2} \mathbf{F}(t, \mu_t) \geq \int_M \left(\mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - \frac{d^2}{dt^2} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \right) d\mu(\mathbf{x}).$$

Since $\mathbf{m}_t = e^{-\mathfrak{U}} \text{vol}$, where vol is the Riemannian volume form, it follows from the proof of Proposition 5.1 that $\mathbf{tr} \mathbf{R}_{(x,p)}^t = \text{Rc}_x(p, p)$ and $v(t, x, p) = -\mathfrak{U}(x)$.

Therefore, we obtain

$$\frac{d^2}{dt^2} v(t, \Phi_t(x, p)) = -\frac{d^2}{dt^2} \mathfrak{U}(\varphi_t(x)) = -\nabla^2 \mathfrak{U}(\varphi_t(x)).$$

Let \exp be the Riemannian exponential map and note that

$$\exp(T\nabla \mathbf{f}(\mathbf{x})) = \pi(\Phi_T(d\mathbf{f}_{\mathbf{x}})) = \varphi_T(\mathbf{x}).$$

It follows that

$$d(\mathbf{x}, \varphi_T(\mathbf{x})) = T|\nabla \mathbf{f}_{\mathbf{x}}|.$$

By the above considerations and (9), we have

$$\begin{aligned} \frac{d^2}{dt^2} \mathbf{F}(t, \mu_t) & \geq K \int_M |\Phi_t(\mathbf{x}, d\mathbf{f})|^2 d\mu(\mathbf{x}) \\ & = \frac{K}{T^2} \int_M d^2(\mathbf{x}, \varphi_T(\mathbf{x})) d\mu(\mathbf{x}) \\ & = \frac{2K}{T} C_T(\mu, \nu) \end{aligned}$$

as claimed.

For the converse, let us first proof the case when t_1 is close enough to t_0 . This argument works for a general Hamiltonian system. By the proof of Theorem 2.2 and the above discussion, we have

$$\begin{aligned} (34) \quad \frac{d^2}{dt^2} \mathbf{F}(t, \mu_t) & = \int_M \left(|\nabla^2 u(\varphi_t(x))|^2 \right. \\ & \quad \left. + (\text{Rc} + \nabla^2 \mathfrak{U})_{\varphi_t(x)}(\nabla u, \nabla u) \right) d\mu(\mathbf{x}). \end{aligned}$$

Assume that there is a point \mathbf{x} such that $\text{Rc}_{\mathbf{x}} + \nabla^2 \mathfrak{U}_{\mathbf{x}} < Kg_{\mathbf{x}}$. Let \mathbf{f} be a function such that $\nabla^2 \mathbf{f}(\mathbf{x}) = 0$. Then the following holds at \mathbf{x} and hence in a neighborhood \mathcal{O} around \mathbf{x}

$$|\nabla^2 \mathbf{f}|_x^2 + (\text{Rc} + \nabla^2 \mathfrak{U})_x(\nabla \mathbf{f}, \nabla \mathbf{f}) < K|\nabla \mathbf{f}|_x^2.$$

Let u be a smooth solution of (23) defined on $[0, T] \times M$. By Proposition 4.2, u exists if we assume that $T > 0$ is small enough. Therefore, on a smaller neighborhood, still denoted by \mathcal{O} , and for T small enough, we have

$$|\nabla^2 u|_{\varphi_t(x)}^2 + (\text{Rc} + \nabla^2 \mathfrak{U})_{\varphi_t(x)}(\nabla u, \nabla u) < K|\nabla u|_{\varphi_t(x)}^2$$

for all t in $[0, T]$.

Finally, we let μ be a measure which is supported in \mathcal{O} . Then this together with (34) gives a contradiction. For T not necessarily small, we use the fact that the Hamiltonian is homogeneous in the p -variable. Indeed, let us use the above notation with T replaced by τ . Then we have

$$(35) \quad |\nabla^2 u|_{\varphi_t(x)}^2 + (\text{Rc} + \nabla^2 \mathfrak{U})_{\varphi_t(x)}(\nabla u, \nabla u) < K|\nabla u|_{\varphi_t(x)}^2$$

holds for all x in \mathcal{O} and for all t in $[0, \tau]$.

Let

$$\bar{u}(t, x) = \frac{\tau}{T} u\left(\frac{t\tau}{T}, x\right).$$

Clearly, \bar{u} is also a smooth solution of (23) with initial condition

$$\bar{u}(0, \cdot) = \frac{\tau}{T} \mathbf{f}(\cdot).$$

On the other hand, if $\bar{\varphi}_t$ is the optimal map with potential $\frac{\tau}{T} \mathbf{f}$, then

$$(36) \quad \begin{aligned} \bar{\varphi}_t(x) &= \exp\left(\frac{t\tau}{T} \nabla \mathbf{f}\right) \\ &= \varphi_{\frac{t\tau}{T}}(x). \end{aligned}$$

Therefore, after multiplying (35) by a constant $(\frac{\tau}{T})^2$ and replacing t by $\frac{t\tau}{T}$, we have

$$|\nabla^2 \bar{u}|_{\bar{\varphi}_t(x)}^2 + (\text{Rc} + \nabla^2 \mathfrak{U})_{\bar{\varphi}_t(x)}(\nabla \bar{u}, \nabla \bar{u}) < K|\nabla \bar{u}|_{\bar{\varphi}_t(x)}^2$$

for all x in \mathcal{O} and all t in $[0, T]$.

Again, by letting μ be a measure supported in \mathcal{O} , we get a contradiction. \square

Remark 7.2. We remark that the first half of the proof of Corollary 2.3 (i.e. the case where T is small enough) works for any Hamiltonian \mathbf{H} . The second half of the proof (i.e. the general case) works for

Hamiltonians which are homogeneous in the p -variable. In particular, it works for the Finsler Hamiltonians. We also remark that (36) also follows from the fact that \mathbf{H} is homogeneous in the p -variable (see the proof of [2, Lemma 5]).

8. PROOF OF COROLLARY 2.4 AND 2.7

This section is devoted to the proof of Corollary 2.4 and 2.7.

Proof of Corollary 2.4. From the proof of Theorem 6.1, we see that $\pi^* \text{vol}_{(x,p)}(f_1^{t,t}, \dots, f_n^{t,t}) = 1$. Assume $k = k(t)$ depends only on t and u satisfies the Hamilton-Jacobi equation $\dot{u} + \frac{1}{2}|\nabla u|^2 + U = 0$ with initial condition $u_{t_0} = \mathbf{f}$. Then

$$\begin{aligned}
 & \frac{d}{dt} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \\
 (37) \quad &= \frac{d}{dt} (k(t)u(t, \varphi_t(\mathbf{x}))) \\
 &= k(t)\dot{u}(t, \varphi_t(\mathbf{x})) + k(t) \langle \nabla u(t, \varphi_t(\mathbf{x})), \dot{\varphi}_t(\mathbf{x}) \rangle + \dot{k}(t)u(t, \varphi_t(\mathbf{x})) \\
 &= k(t) \left(\frac{1}{2} |\nabla u(t, \varphi_t(\mathbf{x}))|_{\varphi_t(\mathbf{x})}^2 - U(t, \varphi_t(\mathbf{x})) \right) + \dot{k}(t)u(t, \varphi_t(\mathbf{x})).
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{d^2}{dt^2} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) = \frac{d^2}{dt^2} (k(t)u(t, \varphi_t(\mathbf{x}))) \\
 (38) \quad &= -k(t) \left(\dot{U}(t, \varphi_t(\mathbf{x})) + 2 \langle \nabla U(t, \varphi_t(\mathbf{x})), \nabla u(t, \varphi_t(\mathbf{x})) \rangle \right) \\
 &+ \frac{c_1(t)}{2} \text{Rc}(\nabla u(t, \varphi_t(\mathbf{x})), \nabla u(t, \varphi_t(\mathbf{x}))) + \frac{c_2(t)}{2} |\nabla u(t, \varphi_t(\mathbf{x}))|^2 \\
 &+ 2\dot{k}(t) \left(\frac{1}{2} |\nabla u(t, \varphi_t(\mathbf{x}))|^2 - U(t, \varphi_t(\mathbf{x})) \right) + \ddot{k}(t)u(t, \varphi_t(\mathbf{x})).
 \end{aligned}$$

If we compare (38) and Theorem 6.1, then we can get rid of the term $\text{Rc}(\nabla u, \nabla u)$ by setting $c_1 k = -2$ and get rid of the term $\langle \nabla R, \nabla u \rangle$ in Theorem 6.1 by setting $U = -\frac{1}{2k^2}R$. Therefore, (38), (33), and Theorem 6.1 together gives

$$\begin{aligned}
 (39) \quad & \text{tr}(\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t) - \frac{d^2}{dt^2} v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) = -\frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4} - \ddot{k}(t)u(t, \varphi_t(\mathbf{x})) \\
 &+ \left(c_2(t)k(t) - 2\dot{k}(t) \right) \left(\frac{1}{2} |\nabla u(t, \varphi_t(\mathbf{x}))|^2 + \frac{1}{2k(t)^2} R(t, \varphi_t(\mathbf{x})) \right)
 \end{aligned}$$

and

$$(40) \quad \begin{aligned} \frac{d}{dt}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) &= \dot{k}(t)u(t, \varphi_t(\mathbf{x})) \\ &+ k(t) \left(\frac{1}{2}|\nabla u(t, \varphi_t(\mathbf{x}))|^2 + \frac{1}{2k(t)^2}R(t, \varphi_t(\mathbf{x})) \right). \end{aligned}$$

Next, we compare (39) and (40). We get rid of the term $u(t, \varphi_t(\mathbf{x}))$ in both expressions by choosing $\dot{k} + b\dot{k} = 0$ and we obtain

$$\begin{aligned} \mathbf{tr}(\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t) - \frac{d^2}{dt^2}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) - b(t)\frac{d}{dt}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) &= -\frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4} \\ &+ \left(c_2(t)k(t) - 2\dot{k}(t) - b(t)k(t) \right) \left(\frac{1}{2}|\nabla u(t, \varphi_t(\mathbf{x}))|^2 + \frac{1}{2k(t)^2}R(t, \varphi_t(\mathbf{x})) \right). \end{aligned}$$

Finally, if we choose $c_2(t)k(t) - 2\dot{k}(t) - b(t)k(t) = 0$, then

$$\mathbf{tr}(\mathbf{R}_{(\mathbf{x}, \mathbf{p})}^t) - \frac{d^2}{dt^2}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) - b(t)\frac{d}{dt}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) = -\frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4}.$$

□

Proof of Corollary 2.7. Let \bar{g} be a solution of the Ricci flow $\dot{\bar{g}} = 2\text{Rc}$ and let $g = \sqrt{t}\bar{g}$ (Note that the Ricci curvatures of g and \bar{g} are the same). Let vol and $\overline{\text{vol}}$ be the volume forms of g and \bar{g} , respectively.

The functional $\bar{\mathbf{F}}$ is defined by

$$\begin{aligned} \bar{\mathbf{F}}(t, \mu) &= \int_M (\log \bar{\rho} + t^{-1/2}u) d\mu + \frac{n}{2} \log t \\ &= \int_M \log(\bar{\rho} e^{t^{-1/2}u}) d\mu + \frac{n}{2} \log t, \end{aligned}$$

where $\mu = \bar{\rho} \overline{\text{vol}}$.

Since $\text{vol} = t^{n/4} \overline{\text{vol}}$, we have

$$\rho e^{-t^{-1/2}u} \text{vol} = \rho \mathbf{m}_t = \mu = \bar{\rho} \overline{\text{vol}} = t^{-n/4} \bar{\rho} \text{vol}.$$

It follows that

$$\begin{aligned} \bar{\mathbf{F}}(t, \mu) &= \int_M \log(t^{n/4} \rho) d\mu + \frac{n}{2} \log t \\ &= \mathbf{F}(t, \mu) + \frac{3n}{4} \log t \end{aligned}$$

Let $m = -\frac{1}{2}$ and let $C = -1$. Then Theorem 2.5 gives

$$\begin{aligned} \frac{d^2}{dt^2} \bar{\mathbf{F}}(t, \mu_t) + \frac{3}{2t} \frac{d}{dt} \bar{\mathbf{F}}(t, \mu_t) \\ = \frac{d^2}{dt^2} \mathbf{F}(t, \mu_t) + \frac{3}{2t} \frac{d}{dt} \mathbf{F}(t, \mu_t) + \frac{3n}{8t^2} \geq 0. \end{aligned}$$

□

9. PROOF OF THEOREM 2.10

Proof of Theorem 2.10. We use the notations in Lemma 7.1. By Lemma 7.1, we have

$$\frac{d}{dt}\mathbf{F}(t, \mu_t) = - \int_M q(r_{\mathbf{x}}^t)^q \left(\frac{d}{dt}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) + \mathbf{tr}(S(t)) \right) d\mu(\mathbf{x})$$

and

$$\begin{aligned} \frac{d^2}{dt^2}\mathbf{F}(t, \mu_t) &= \int_M (r_{\mathbf{x}}^t)^q \left[q^2 \left(\frac{d}{dt}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) + \mathbf{tr}(S(t)) \right)^2 \right. \\ &\quad \left. + q \left(\mathbf{tr}(S(t)^2) + \mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - \frac{d^2}{dt^2}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \right) \right] d\mu(\mathbf{x}) \\ &\geq \int_M (r_{\mathbf{x}}^t)^q \left[q^2 b_1(t) \left(\frac{d}{dt}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) + \mathbf{tr}(S(t)) \right) - \frac{q^2 b_1(t)^2}{4} \right. \\ &\quad \left. + q \left(b_2(t) \mathbf{tr}(S(t)) - \frac{nb_2(t)^2}{4} + \mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - \frac{d^2}{dt^2}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \right) \right] d\mu(\mathbf{x}). \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{d^2}{dt^2}\mathbf{F}(t, \mu_t) + (qb_1(t) + b_2(t)) \frac{d}{dt}\mathbf{F}(t, \mu_t) \\ &\geq \int_M q(r_{\mathbf{x}}^t)^q \left[\mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - b_2(t) \frac{d}{dt}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \right. \\ &\quad \left. - \frac{d^2}{dt^2}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) - \frac{qb_1(t)^2}{4} - \frac{nb_2(t)^2}{4} \right] d\mu(\mathbf{x}). \end{aligned}$$

□

Proof of Corollary 2.12. The way to find the appropriate unknown functions (for instance c_1, c_2, U) is in the proof of Corollary 2.4. Here we simply prove the result directly. But this follows immediately from (39), (40), and Theorem 2.10 since

$$\begin{aligned} &\mathbf{tr}(\mathbf{R}_{\Phi_t(\mathbf{x}, d\mathbf{f})}^t) - b_2(t) \frac{d}{dt}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) - \frac{d^2}{dt^2}v(t, \Phi_t(\mathbf{x}, d\mathbf{f})) \\ &= -\frac{nc_2(t)}{2} - \frac{nc_2(t)^2}{4}. \end{aligned}$$

□

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E-mail address: wylee@math.cuhk.edu.hk

ROOM 216, LADY SHAW BUILDING, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG